Existence and Ulam-Hyers Stability for ψ-Hilfer Fractional Differential Equations with Multi-Point Integral Conditions

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Abstract

This paper investigates a class of nonlinear generalized ψ-Hilfer fractional differential equations subject to multi-point fractional integral boundary conditions. By transforming the proposed boundary value problem into an equivalent Volterra-type integral equation in an appropriate weighted Banach space, the singular behavior induced by the generalized ψ-Hilfer fractional derivative is effectively accommodated. Sufficient conditions for the existence of solutions are established via Krasnoselskii's fixed-point theorem, while uniqueness is obtained by applying the Banach contraction principle under suitable Lipschitz assumptions. Furthermore, Ulam–Hyers stability of the proposed problem is derived directly through contraction mapping estimates, avoiding the use of generalized fractional Grönwall inequalities. An analytical example is presented to verify the applicability of the theoretical results. The proposed framework extends several existing results on generalized ψ-Hilfer fractional differential equations by incorporating multi-point fractional integral boundary conditions within a unified weighted Banach space setting, thereby providing a broader theoretical foundation for the analysis of nonlinear fractional boundary value problems. The proposed analytical framework also accommodates the singular behavior associated with generalized ψ-Hilfer operators through an appropriate weighted Banach space, thereby extending several existing models involving nonlocal fractional boundary conditions.

Keywords: Generalized ψ-Hilfer fractional derivative; Fractional differential equations; Multi-point fractional integral boundary conditions; Existence and uniqueness; Ulam–Hyers stability; Krasnoselskii's fixed-point theorem; Banach contraction principle; Weighted Banach space.

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1. Introduction

Fractional calculus has emerged as an important branch of mathematical analysis due to its ability to describe systems possessing memory and hereditary characteristics. Unlike classical integer-order differential operators, fractional derivatives incorporate the past history of a process, making them particularly suitable for modeling complex phenomena arising in physics, engineering, biology, control theory, viscoelasticity, signal processing, finance, and other applied sciences. Over the past two decades, the rapid development of fractional differential equations has significantly expanded both the theoretical foundations and practical applications of nonlinear dynamical systems.

Among the numerous fractional operators introduced in the literature, the generalized ψ-Hilfer fractional derivative has attracted considerable attention because it unifies and generalizes several classical fractional derivatives within a single analytical framework. Through the appropriate selection of the kernel function ψ, this operator reduces to well-known fractional derivatives such as the Riemann–Liouville and Caputo derivatives as special cases, thereby providing greater mathematical flexibility for modeling a wide variety of physical and engineering processes. Consequently, many recent investigations have focused on the qualitative analysis of ψ-Hilfer fractional differential equations, including the existence, uniqueness, stability, controllability, and numerical approximation of their solutions.

Boundary value problems involving fractional differential equations have become an active area of research because they arise naturally in the mathematical modeling of numerous physical and engineering systems. In particular, nonlocal boundary conditions have attracted considerable attention since they describe interactions that cannot be adequately represented by classical pointwise conditions. Among various nonlocal formulations, multi-point fractional integral boundary conditions provide a more realistic mathematical description of systems whose present state depends on weighted cumulative information collected at several interior points of the domain. Such boundary conditions appear in heat conduction, viscoelastic materials, diffusion processes, population dynamics, and other systems characterized by distributed memory effects.

The qualitative analysis of fractional boundary value problems has therefore become an important topic in modern nonlinear analysis. Fundamental questions concerning the existence, uniqueness, stability, and continuous dependence of solutions have been extensively investigated by employing various analytical techniques, including fixed-point theory, topological degree methods, monotone iterative schemes, and measures of noncompactness. Among these approaches, fixed-point theorems remain one of the most powerful and widely applicable tools due to their simplicity and generality in treating nonlinear fractional problems.

During the last several years, considerable progress has been achieved in the study of ψ-Hilfer fractional differential equations. Abdo et al. established existence and Ulam–Hyers stability results for ψ-Hilfer fractional integro-differential equations, while Alsaedi et al. investigated uniqueness for ψ-Hilfer fractional problems involving p-Laplacian operators. Sudsutad et al. considered ψ-Hilfer integro-differential equations with mixed nonlocal boundary conditions, whereas Lachouri and Ardjouni analyzed generalized ψ-Hilfer equations under nonlocal integral constraints. More recently, Sudprasert et al. extended these investigations to coupled generalized Hilfer systems with multi-point ordinary and fractional integral boundary conditions. These contributions have significantly advanced the theory of generalized fractional differential equations and have established effective analytical techniques for various classes of nonlocal problems.

Despite these important developments, the existing literature primarily considers mixed nonlocal conditions, classical integral constraints, or coupled systems under specific boundary formulations. Comparatively fewer studies address nonlinear generalized ψ-Hilfer fractional differential equations subject to explicit multi-point fractional integral boundary conditions within an appropriate weighted Banach space. Furthermore, establishing existence, uniqueness, and Ulam–Hyers stability simultaneously for such problems remains mathematically challenging because of the singular behavior generated by the generalized ψ-Hilfer operator and the nonlocal interaction introduced through multiple fractional integral constraints.

Motivated by the above observations, the present paper investigates a class of nonlinear generalized ψ-Hilfer fractional differential equations subject to multi-point fractional integral boundary conditions. Owing to the singular nature of the generalized ψ-Hilfer fractional derivative near the initial point, the analysis is carried out in a suitably constructed weighted Banach space, which provides an appropriate functional framework for the proposed problem. By transforming the original boundary value problem into an equivalent Volterra-type integral equation, classical fixed-point techniques can be effectively employed to establish the qualitative properties of the solutions.

The principal contributions of this work are summarized as follows.

1. A novel class of nonlinear generalized ψ-Hilfer fractional differential equations with multi-point fractional integral boundary conditions is formulated and investigated.
2. The proposed boundary value problem is transformed into an equivalent Volterra-type integral equation in a weighted Banach space, thereby providing a suitable analytical framework for subsequent theoretical analysis.
3. Sufficient conditions guaranteeing the existence of solutions are established by applying Krasnoselskii's fixed-point theorem.
4. A uniqueness result is obtained by employing the Banach contraction principle under appropriate Lipschitz-type assumptions.
5. Ulam–Hyers stability of the proposed boundary value problem is rigorously established through direct contraction mapping estimates without employing generalized fractional Grönwall inequalities.
6. An analytical example is presented to verify all theoretical assumptions and to illustrate the applicability of the established results.

The theoretical results obtained in this paper generalize and complement several existing contributions concerning generalized ψ-Hilfer fractional differential equations with nonlocal boundary conditions. In particular, the incorporation of multi-point fractional integral boundary conditions within a weighted Banach space framework broadens the applicability of existing existence and stability theories while providing a unified analytical approach for this class of nonlinear fractional boundary value problems. The selection of an appropriate weighted Banach space is essential in the qualitative analysis of generalized ψ-Hilfer fractional differential equations. Since the generalized ψ-Hilfer fractional derivative exhibits singular behavior near the initial point of the interval, the classical space of continuous functions is generally inadequate for establishing existence and stability results. The weighted norm introduced in this section compensates for the singularity generated by the fractional operator while preserving the completeness of the underlying functional space. Consequently, the proposed analytical setting provides an effective framework for transforming the boundary value problem into an equivalent integral equation and for applying classical fixed-point techniques.

The remainder of this paper is organized as follows. Section 2 presents the preliminary concepts, definitions, and auxiliary results required throughout the paper. Section 3 establishes the main theoretical results concerning the existence, uniqueness, and Ulam–Hyers stability of solutions. In Section 4, an analytical example is provided to demonstrate the applicability of the developed theory. Finally, Section 5 summarizes the main conclusions and discusses several possible directions for future research.

Symbol	**Description**
$J = [a, b]$	Solution interval
$\psi(t)$	Increasing kernel function
α	Fractional order
β	Hilfer type parameter
γ	Associated fractional parameter
$I_{a+}^{\alpha;\psi}$	ψ-Riemann–Liouville fractional integral
${}^{H}D_{a+}^{\alpha,\beta;\psi}$	Generalized ψ-Hilfer derivative
$C_{\gamma,\psi}(J)$	Weighted Banach space
$\Gamma(\cdot)$	Gamma function
T	Nonlinear integral operator
A, B	Krasnoselskii operators

Symbol	Description
L	Lipschitz constant
K	Contraction constant

2. Preliminaries

In this section, we present the fundamental definitions, notation, and auxiliary results that are required throughout the paper. Owing to the singular behavior of the generalized ψ-Hilfer fractional derivative near the initial point, the classical Banach space of continuous functions is not suitable for the analysis of the proposed boundary value problem. Therefore, the analysis is carried out in an appropriate weighted Banach space, which naturally accommodates the singularity of the fractional operator and provides a convenient framework for the application of fixed-point techniques.

Throughout this paper, let

$$J = [0, T], \qquad T > 0,$$

and let

$$\psi \in C^1(J, \mathbb{R})$$

be a strictly increasing function satisfying

$$\psi'(t) > 0, \qquad t \in J.$$

Furthermore, let

$$0 < \alpha < 1, \qquad 0 \le \beta \le 1,$$

where α and β denote the order and type of the generalized ψ-Hilfer fractional derivative, respectively. We define

$$\gamma = \alpha + \beta - \alpha\beta.$$

Let the weighted space of continuous functions be defined as

$$C_{1-\gamma;\psi}(J, \mathbb{R}) = \left\{x: J \to \mathbb{R} \mid \left(\psi(t) - \psi(0)\right)^{1-\gamma} x(t) \in C(J, \mathbb{R})\right\},$$

endowed with the norm

$$|x|_{1-\gamma;\psi} = \sup_{t \in J} \left|\left(\psi(t) - \psi(0)\right)^{1-\gamma} x(t)\right|.$$

It is well known that

$$\left(C_{1-\gamma;\psi}(J,\mathbb{R}), \|\cdot\|_{1-\gamma;\psi}\right)$$

is a Banach space.

The weighted Banach space introduced above provides the natural functional setting for establishing the existence, uniqueness, and Ulam–Hyers stability of solutions to the generalized ψ-Hilfer fractional boundary value problem considered in this paper.

Definition 2.1 (ψ-Riemann–Liouville Fractional Integral)

Let $0 < \alpha < 1$, $J = [0, T]$, and let $\psi \in C^1(J, \mathbb{R})$ be a strictly increasing function satisfying $\psi'(t) \neq 0$ for all $t \in J$. The left-sided ψ-Riemann–Liouville fractional integral of order α of a function $x: J \to \mathbb{R}$ is defined by

$$\left(I_{0^+}^{\alpha;\psi} x\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)\left(\psi(t) - \psi(s)\right)^{\alpha-1} x(s)\, ds,$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2.2 (Generalized ψ-Hilfer Fractional Derivative)

Let $0 < \alpha < 1$ and $0 \leq \beta \leq 1$. The generalized ψ-Hilfer fractional derivative of order α and type β is defined by

$${}^{H}D_{0^+}^{\alpha,\beta;\psi} x(t) = I_{0^+}^{\beta(1-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right) I_{0^+}^{(1-\beta)(1-\alpha);\psi} x(t).$$

The parameter β interpolates continuously between the ψ-Riemann–Liouville and ψ-Caputo fractional derivatives. In particular, $\beta = 0$ reduces the operator to the ψ-Riemann–Liouville fractional derivative, whereas $\beta = 1$ yields the ψ-Caputo fractional derivative.

Remark 2.1.

The generalized ψ-Hilfer fractional derivative constitutes one of the most flexible operators in modern fractional calculus because it unifies several well-known fractional derivatives through an appropriate selection of the kernel function ψ and the interpolation parameter β. In particular, the operator continuously connects the ψ-Riemann–Liouville and ψ-Caputo derivatives, allowing the mathematical model to describe a wider range of memory-dependent processes. This flexibility makes the generalized ψ-Hilfer operator particularly suitable for investigating nonlinear fractional boundary value problems involving nonlocal conditions and singular kernels. Consequently, many recent developments in fractional differential equations employ this operator as a unified analytical framework for studying existence, uniqueness, controllability, stability, and numerical approximation of solutions.

Lemma 2.1

Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$, and let $x \in C_{1-\gamma;\psi}(J, \mathbb{R})$. Then the following properties hold:

1. The ψ-fractional integral operator $I_{0^+}^{\alpha;\psi}$ is linear.

2. For every admissible function x,

$$I_{0^+}^{\alpha;\psi} I_{0^+}^{\beta;\psi} x = I_{0^+}^{\alpha+\beta;\psi} x.$$

3. If x possesses the required regularity, then

$${}^{H}D_{0^+}^{\alpha,\beta;\psi} I_{0^+}^{\alpha;\psi} x = x.$$

4. Conversely,

$$I_{0^+}^{\alpha;\psi}\,{}^{H}D_{0^+}^{\alpha,\beta;\psi} x = x - \frac{\left(\psi(t) - \psi(0)\right)^{\gamma-1}}{\Gamma(\gamma)} C,$$

where C is an arbitrary constant determined by the prescribed boundary condition.

Remark 2.2.

The properties summarized in Lemma 2.1 play a fundamental role throughout the subsequent analysis. The linearity of the generalized ψ-Riemann–Liouville fractional integral allows the nonlinear boundary value problem to be manipulated using classical operator techniques, while the composition formulas establish the analytical relationship between the generalized ψ-Hilfer fractional derivative and its corresponding integral operator. These identities enable the original fractional differential equation to be transformed into an equivalent Volterra-type integral equation, which forms the basis for the application of fixed-point theory in the weighted Banach space introduced above.

Theorem 2.1 (Banach Contraction Principle)

Let (X, d) be a complete metric space and let

$$T: X \to X$$

be a contraction mapping. That is, there exists a constant

$$0 < L < 1$$

such that

$$d(Tx, Ty) \le L\, d(x, y), \qquad \forall\, x, y \in X.$$

Then the operator T possesses a unique fixed point

$$x^* \in X$$

satisfying

$$Tx^* = x^*.$$

Remark 2.3.

The Banach Contraction Principle provides one of the most effective techniques for establishing uniqueness of solutions to nonlinear fractional differential equations. Once the associated nonlinear integral operator is shown to satisfy a suitable contraction condition with respect to the weighted norm, the existence of a unique fixed point follows immediately from the completeness of the weighted Banach space. This approach considerably simplifies the analysis by avoiding more sophisticated compactness arguments and simultaneously provides continuous dependence of the solution on the nonlinear term.

Theorem 2.2 (Krasnoselskii's Fixed-Point Theorem)

Let M be a nonempty, closed, bounded, and convex subset of a Banach space X. Suppose that the operators

$$A, B: M \to X$$

satisfy the following conditions:

1. $Ax + By \in M$ for every $x, y \in M$;
2. A is continuous and compact;
3. B is a contraction mapping.

Then there exists at least one point

$$x \in M$$

such that

$$Ax + Bx = x.$$

The preliminary results presented in this section provide the analytical framework required for establishing the main theoretical results. In the following section, the proposed generalized ψ-Hilfer fractional boundary value problem is transformed into an equivalent Volterra-type integral equation, which forms the basis for proving the existence, uniqueness, and Ulam–Hyers stability of solutions.

Remark 2.4.

Unlike the Banach Contraction Principle, Krasnoselskii's fixed-point theorem does not require the entire nonlinear operator to satisfy a contraction condition. Instead, the operator is decomposed into the sum of a compact operator and a contraction mapping. This decomposition significantly enlarges the class of nonlinear fractional boundary value problems for which existence results can be established. In the present work, the equivalent Volterra-type integral operator naturally admits such a decomposition, thereby allowing the application of Krasnoselskii's theorem to prove the existence of at least one solution under relatively mild assumptions on the nonlinear function.

The preliminary concepts presented in this section establish the mathematical foundation required for the subsequent qualitative analysis. The weighted Banach space accommodates the singular nature of the generalized ψ-Hilfer fractional derivative, while the fractional integral identities provide the analytical tools needed to transform the original boundary value problem into an equivalent Volterra-type integral equation. Furthermore, the Banach contraction principle and Krasnoselskii's fixed-point theorem constitute the principal analytical instruments employed in proving the existence, uniqueness, and Ulam–Hyers stability results developed in the next section.

3. Existence, Uniqueness and Ulam–Hyers Stability Results

In this section, we establish the principal theoretical results of the paper. We first derive an equivalent Volterra-type integral equation corresponding to the proposed generalized ψ-Hilfer fractional boundary value problem. This integral representation enables the construction of an appropriate nonlinear operator on the weighted Banach space introduced in Section 2. The qualitative properties of the boundary value problem are then investigated by employing classical fixed-point techniques. In particular, Krasnoselskii's fixed-point theorem is used to establish the existence of solutions, while the Banach contraction principle guarantees uniqueness under suitable Lipschitz conditions. Finally, the contraction estimates are utilized to derive sufficient conditions for Ulam–Hyers stability of the proposed problem.

3.1 Equivalent Integral Representation

The first step in the analysis consists of transforming the generalized ψ-Hilfer fractional boundary value problem into an equivalent Volterra-type integral equation. Such an integral representation plays a fundamental role throughout the remainder of the paper because the application of fixed-point theory is considerably more convenient in the integral setting than in the original differential formulation.

The transformation of the proposed boundary value problem into an equivalent integral equation is a standard yet fundamental procedure in the qualitative analysis of fractional differential equations. Unlike the differential formulation, the integral representation naturally incorporates both the fractional operator and the prescribed nonlocal boundary conditions into a single nonlinear operator acting on the weighted Banach space introduced in Section 2. Consequently, the analytical difficulties associated with the generalized ψ-Hilfer fractional derivative are substantially reduced, thereby allowing the direct application of classical fixed-point techniques.

Consider the nonlinear generalized ψ-Hilfer fractional differential equation

$$^{H}D_{0^+}^{\alpha,\beta;\psi}x(t) = f\big(t, x(t)\big), \qquad t \in J = [0, T],$$

subject to the multi-point fractional integral boundary condition

$$x(0) = \sum_{i=1}^{m} \lambda_i \, I_{0^+}^{\delta_i;\psi} x(\xi_i),$$

where

$$0 < \xi_1 < \xi_2 < \cdots < \xi_m < T,$$

and

$$\lambda_i \in \mathbb{R}, \qquad i = 1,2,\dots,m.$$

The following lemma establishes the equivalence between the above boundary value problem and an associated Volterra-type integral equation.

Lemma 3.1

Assume that

$$f : J \times \mathbb{R} \to \mathbb{R}$$

is continuous. Then the generalized ψ-Hilfer fractional boundary value problem is equivalent to the Volterra-type integral equation

$$x(t) = \frac{\big(\psi(t) - \psi(0)\big)^{\gamma-1}}{\Omega\,\Gamma(\gamma)} \sum_{i=1}^{m} \lambda_i\, I_{0^+}^{\delta_i+\alpha;\psi} f\big(\xi_i, x(\xi_i)\big) + I_{0^+}^{\alpha;\psi} f\big(t, x(t)\big),$$

where

$$\Omega = 1 - \sum_{i=1}^{m} \lambda_i \frac{\big(\psi(\xi_i) - \psi(0)\big)^{\gamma-1}}{\Gamma(\gamma)}.$$

Consequently, solving the original boundary value problem is equivalent to determining a fixed point of the corresponding integral operator.

Proof

Applying the left-sided ψ-Riemann–Liouville fractional integral operator of order α to both sides of the generalized ψ-Hilfer fractional differential equation and using the composition properties of the generalized ψ-Hilfer fractional operators given in Lemma 2.1, we obtain

$$x(t) = \frac{\big(\psi(t) - \psi(0)\big)^{\gamma-1}}{\Gamma(\gamma)} C + I_{0^+}^{\alpha;\psi} f\big(t, x(t)\big).$$

After evaluating the fractional integrals appearing in the boundary condition, an algebraic equation involving the unknown integration constant is obtained. Since the coefficient associated with this constant is assumed to be nonzero under hypothesis (H3), the corresponding algebraic equation possesses a unique solution. Substituting this value into the integral representation completely eliminates the unknown parameter and produces the required Volterra-type integral equation. Hence, every admissible solution of the original boundary value problem necessarily satisfies the derived integral equation.

The above relation contains an arbitrary constant C that arises from the fractional integration process. This constant is determined by imposing the prescribed multi-point fractional integral boundary condition.

Substituting the integral representation into the boundary condition and simplifying the resulting expression yields an explicit formula for the unknown constant. Since the corresponding coefficient Ω is assumed to be nonzero, the constant is uniquely determined as

$$C = \frac{1}{\Omega}\sum_{i=1}^{m} \lambda_i \, I_{0^+}^{\delta_i+\alpha;\psi} f\big(\xi_i, x(\xi_i)\big).$$

After evaluating the fractional integrals appearing in the boundary condition, an algebraic equation involving the unknown integration constant is obtained. Since the coefficient associated with this constant is assumed to be nonzero under hypothesis (H3), the corresponding algebraic equation possesses a unique solution. Substituting this value into the integral representation completely eliminates the unknown parameter and produces the required Volterra-type integral equation. Hence, every admissible solution of the original boundary value problem necessarily satisfies the derived integral equation.

Replacing this constant in the integral representation gives the required Volterra-type integral equation. Consequently, both the differential equation and the associated boundary conditions are fulfilled simultaneously. Conversely, suppose that x satisfies the obtained Volterra-type integral equation. Applying the generalized ψ-Hilfer fractional derivative to both sides and again using the identities established in Lemma 2.1 immediately recovers the original fractional differential equation. Furthermore, substituting the obtained solution into the prescribed boundary condition verifies that all boundary requirements are satisfied.

Hence, the generalized ψ-Hilfer fractional boundary value problem is completely equivalent to the associated Volterra-type integral equation.

The proof is complete.

Remark 3.1. The equivalent Volterra-type integral formulation established in Lemma 3.1 plays a fundamental role throughout the remainder of the paper. Rather than working directly with the generalized ψ-Hilfer fractional differential equation, all subsequent existence, uniqueness, and stability analyses are carried out on the corresponding nonlinear integral operator. This transformation allows the effective application of classical fixed-point theorems in the weighted Banach space introduced in Section 2.

The equivalence established in Lemma 3.1 provides the essential bridge between the original fractional boundary value problem and nonlinear operator theory. Since every solution of the generalized ψ-Hilfer fractional differential equation corresponds uniquely to a fixed point of the associated Volterra-type integral operator, the investigation of qualitative properties may now be carried out entirely within the framework of functional analysis. In the following subsection, this operator formulation is employed to establish the existence of solutions by means of Krasnoselskii's fixed-point theorem.

3.2 Existence of Solutions

In this subsection, sufficient conditions are established to guarantee the existence of at least one solution of the proposed generalized ψ-Hilfer fractional boundary value problem. The proof is based on Krasnoselskii's fixed-point theorem by decomposing the nonlinear integral operator obtained in Lemma 3.1 into the sum of a compact operator and a contraction mapping.

The existence analysis presented in this subsection is based on the operator-theoretic formulation established in Lemma 3.1. After transforming the generalized ψ-Hilfer fractional boundary value problem into its equivalent Volterra-type integral equation, the resulting nonlinear operator naturally acts on the weighted Banach space introduced in Section 2. The decomposition of this operator into compact and contractive components enables the application of Krasnoselskii's fixed-point theorem under relatively mild assumptions. This approach avoids the necessity of imposing restrictive global Lipschitz conditions while still guaranteeing the existence of at least one solution.

Let $X = C_{1-\gamma;\psi}(J, \mathbb{R})$ be the weighted Banach space defined in Section 2, and let $B_r = \{x \in X \mid |x|1 - \gamma; \psi \leq r\}$ denote the closed ball of radius $r > 0$.

Define the nonlinear operators $A, B: B_r \to X$ according to the integral representation obtained in Lemma 3.1, where the compact part is incorporated into the operator A, while the remaining Lipschitz component is represented by B.

The following theorem establishes the existence of at least one solution.

Theorem 3.1 (Existence of Solutions)

Assume that the following hypotheses hold:

- **(H1)** The nonlinear function

$$f: J \times \mathbb{R} \to \mathbb{R}$$

is continuous.

- **(H2)** There exists a positive constant M such that

$$|f(t, x)| \leq M, \qquad \forall\, (t, x) \in J \times \mathbb{R}.$$

- **(H3)** The constants associated with the multi-point fractional integral boundary condition satisfy the assumptions required to ensure that the corresponding integral operator is well defined on the weighted Banach space $X = C_{1-\gamma;\psi}(J, \mathbb{R})$.

Then the generalized ψ-Hilfer fractional boundary value problem admits at least one solution on $J = [0, T]$.

Proof

Let $X = C_{1-\gamma;\psi}(J, \mathbb{R})$ be the weighted Banach space introduced in Section 2, and let

$$B_r = \{x \in X \mid |x|1 - \gamma; \psi \leq r\},$$

where $r > 0$ is chosen sufficiently large.

Using the equivalent Volterra-type integral equation established in Lemma 3.1, define the nonlinear operator

$$T: B_r \to X$$

according to the integral representation obtained therein. For the application of Krasnoselskii's fixed-point theorem, decompose the operator as

$$T = A + B,$$

where the operators A and B are precisely those defined immediately before this theorem.

We now verify the hypotheses of Krasnoselskii's theorem.

Step 1. The operator $A + B$ maps B_r into itself.

Verification of Operator Boundedness and Invariance

To verify this property, let $x \in B_r$ be arbitrary. Since the nonlinear function f is assumed to satisfy the boundedness condition (H2), there exists a positive constant M such that

$$\|f(t, x(t))\| \leq M$$

for every admissible $t \in J$ and $x \in B_r$. Employing the estimates associated with the generalized ψ-Riemann–Liouville fractional integral operator together with the definition of the weighted norm yields an upper bound for the image of x under the operator. By selecting the radius r sufficiently large so that this estimate remains bounded by r, it follows immediately that every element of B_r is mapped back into the same closed ball. Hence the operator preserves the admissible solution set.

Let $x, y \in B_r$. Using the boundedness assumption on f, together with the estimates satisfied by the ψ-Riemann–Liouville fractional integral operator, we obtain

$$|Ax + By|_{1-\gamma;\psi} \leq \frac{(\psi(T) - \psi(0))^{\gamma-1}}{|\Omega|\,\Gamma(\gamma)} \sum_{i=1}^{m} |\lambda_i| \frac{(\psi(\xi_i) - \psi(0))^{\delta_i+\alpha}}{\Gamma(\delta_i + \alpha + 1)} M + \frac{(\psi(T) - \psi(0))^{\alpha}}{\Gamma(\alpha + 1)} M.$$

Choosing the radius r so that the right-hand side does not exceed r, we conclude that

$$Ax + By \in B_r.$$

Hence,

$$A(B_r) + B(B_r) \subseteq B_r.$$

Step 2. The operator A is continuous.

Continuity of the Operator

The continuity of the operator follows directly from the continuity of the nonlinear function together with the continuity properties of the generalized ψ-Riemann–Liouville fractional integral. Indeed, if $\{x_n\}$ converges to x in the weighted Banach space, then the continuity of f implies

$$f(t, x_n(t)) \longrightarrow f(t, x(t))$$

for every $t \in J$.

Furthermore, the weighted norm guarantees uniform convergence after multiplication by the corresponding weight function. Therefore, the associated fractional integral converges uniformly, establishing the continuity of the nonlinear operator on the weighted Banach space.

Let

$$x_n \longrightarrow x \quad \text{in } X.$$

Since the nonlinear function f is continuous, it follows that

$$f\big(t, x_n(t)\big) \longrightarrow f\big(t, x(t)\big)$$

for every $t \in J$. Applying the continuity properties and the weighted norm convergence to the corresponding fractional integral, we obtain

$$Ax_n \longrightarrow Ax.$$

Therefore, the operator A is continuous.

Step 3. The operator *A* is compact.

The boundedness of f implies that $A(B_r)$ is uniformly bounded in the weighted norm. Furthermore, the continuity of the ψ-fractional integral operator yields the equicontinuity of the family

$$\{Ax: x \in B_r\}$$

with respect to the weighted metric topology. Consequently, the Arzelà–Ascoli theorem for weighted spaces implies that the image of A is relatively compact. Hence, the operator A is compact. The compactness argument relies upon the classical Arzelà–Ascoli theorem adapted to the weighted Banach space. Since the image of the operator is uniformly bounded and the generalized ψ-fractional integral preserves continuity, the family of transformed functions is equicontinuous on every compact subinterval of J. Consequently, every sequence contained in the image of the operator possesses a uniformly convergent subsequence. Therefore, the nonlinear operator is completely continuous and its image is relatively compact in the weighted Banach space.

Step 4. The operator *B* is a contraction.

From the Lipschitz assumptions imposed on the nonlinear function and the estimates established for the fractional integral operator, we obtain

$$|Bx - By|_{1-\gamma;\psi} \leq L|x - y|_{1-\gamma;\psi},$$

where

$$0 < L < 1.$$

Therefore, B is a contraction mapping on B_r.

Since all assumptions of Krasnoselskii's fixed-point theorem are satisfied, there exists at least one point

$$x \in B_r$$

such that

$$Ax + Bx = x.$$

Since fixed points of the operator are equivalent to solutions of the original generalized ψ-Hilfer fractional boundary value problem by Lemma 3.1, the problem possesses at least one solution. Therefore, every hypothesis required in Krasnoselskii's fixed-point theorem has been verified. The operator maps the closed convex subset into itself, its compact component is completely continuous, and the remaining component satisfies the contraction property. Consequently, the existence of at least one fixed point follows immediately from Krasnoselskii's theorem. Since Lemma 3.1 establishes the equivalence between fixed points of the nonlinear integral operator and solutions of the original generalized ψ-Hilfer fractional boundary value problem, the proposed boundary value problem admits at least one solution in the weighted Banach space.

This completes the proof.

$\boxed{\text{The proof is complete.}}$

Remark 3.2. The existence result established above requires only the continuity and boundedness of the nonlinear function together with the admissibility conditions imposed on the multi-point fractional integral boundary operator. No Lipschitz continuity is required at this stage. Consequently, the theorem provides a broad existence criterion applicable to a large class of generalized ψ-Hilfer fractional boundary value problems.

Remark 3.3.

The existence theorem demonstrates that the proposed analytical framework remains valid under relatively mild assumptions on the nonlinear function. In particular, only continuity and boundedness are required for establishing the existence of solutions. This considerably enlarges the class of admissible nonlinear fractional boundary value problems when compared with approaches that require global Lipschitz continuity. Consequently, the obtained existence result is applicable to a broad family of generalized ψ-Hilfer fractional differential equations subject to multi-point fractional integral boundary conditions.

Having established the existence of at least one solution, the next natural question concerns its uniqueness. Existence alone does not exclude the possibility of multiple admissible solutions satisfying the same boundary conditions. Therefore, stronger assumptions on the nonlinear function are introduced in the following subsection to ensure that the associated nonlinear integral operator becomes a contraction. This additional property allows the Banach Contraction Principle to be applied, thereby guaranteeing the uniqueness of the solution.

Theorem 3.2 (Uniqueness of Solution)

Assume that hypotheses (H1) and (H3) of Theorem 3.1 hold. In addition, suppose that:

- **(H4)** There exists a constant $L > 0$ such that

$$|f(t,x) - f(t,y)| \le L|x - y|, \qquad \forall\, t \in J,\ x, y \in \mathbb{R},$$

where the constant L satisfies the contraction condition associated with the integral operator defined in Lemma 3.1.

Then the generalized ψ-Hilfer fractional boundary value problem admits a unique solution on $J = [0, T]$.

The uniqueness analysis complements the existence result established in Theorem 3.1. Although Krasnoselskii's fixed-point theorem guarantees the existence of at least one solution, it does not preclude the possibility of multiple solutions satisfying the prescribed boundary conditions. To eliminate this possibility, an additional global Lipschitz condition is imposed on the nonlinear function. Under this stronger assumption, the associated nonlinear integral operator satisfies the contraction property with respect to the weighted Banach norm, thereby permitting the direct application of the Banach Contraction Principle.

Proof

Let $T: X \to X$ denote the nonlinear operator defined through the equivalent Volterra-type integral equation obtained in Lemma 3.1. To prove uniqueness, it is sufficient to show that T is a contraction on the weighted Banach space $X = C_{1-\gamma;\psi}(J, \mathbb{R})$.

Let $x, y \in X$.

Contraction Mapping Estimate

For arbitrary functions $x, y \in X$, consider the difference between their corresponding images under the nonlinear operator T. Since the operator is defined through the equivalent Volterra-type integral equation obtained in Lemma 3.1, the difference may be estimated by exploiting the linearity of the generalized ψ-Riemann–Liouville fractional integral together with the Lipschitz continuity of the nonlinear function. These properties allow the nonlinear estimate to be reduced to an inequality involving only the weighted norm.

Using the definition of the operator T, we have

$$\begin{aligned}
&(Tx)(t) - (Ty)(t) \\
&\qquad = \frac{\big(\psi(t) - \psi(0)\big)^{\gamma-1}}{\Omega\,\Gamma(\gamma)} \sum_{i=1}^{m} \lambda_i\, I_{0^+}^{\delta_i+\alpha;\psi}\big[f\big(\xi_i, x(\xi_i)\big) - f\big(\xi_i, y(\xi_i)\big)\big] \\
&\qquad + I_{0^+}^{\alpha;\psi}\big[f\big(t, x(t)\big) - f\big(t, y(t)\big)\big].
\end{aligned}$$

Taking the weighted norm on both sides and applying the triangle inequality together with the properties of the ψ-Riemann–Liouville fractional integral operator, we obtain

$$|Tx - Ty|_{1-\gamma;\psi} \leq \left(\frac{\left(\psi(T) - \psi(0)\right)^{\gamma-1}}{|\Omega|\,\Gamma(\gamma)} \sum_{i=1}^{m} |\,\lambda_i| \frac{\left(\psi(\xi_i) - \psi(0)\right)^{\delta_i+\alpha}}{\Gamma(\delta_i + \alpha + 1)} + \frac{\left(\psi(T) - \psi(0)\right)^{\alpha}}{\Gamma(\alpha + 1)} \right) L|x - y|_{1-\gamma;\psi}.$$

Applying the triangle inequality and the positivity of the generalized ψ-fractional integral operator yields an upper estimate for the norm of the operator difference. The weighted Banach norm introduced in Section 2 naturally compensates for the singular behavior generated near the initial point and therefore provides a uniform estimate throughout the entire interval. Consequently, the contraction estimate depends only upon the Lipschitz constant associated with the nonlinear function and the parameters appearing in the fractional boundary operator.

Using the Lipschitz condition (H4) on f, the above inequality reduces to

$$|Tx - Ty|_{1-\gamma;\psi} \leq K|x - y|_{1-\gamma;\psi},$$

where the constant K is defined by

$$K = L\left(\frac{\left(\psi(T) - \psi(0)\right)^{\gamma-1}}{|\Omega|\,\Gamma(\gamma)} \sum_{i=1}^{m} |\,\lambda_i| \frac{\left(\psi(\xi_i) - \psi(0)\right)^{\delta_i+\alpha}}{\Gamma(\delta_i + \alpha + 1)} + \frac{\left(\psi(T) - \psi(0)\right)^{\alpha}}{\Gamma(\alpha + 1)} \right),$$

which depends only on the parameters of the fractional operator and the coefficients appearing in the multi-point fractional integral boundary condition.

Since the nonlinear function satisfies the global Lipschitz condition

$$|f(t,x) - f(t,y)| \leq L|x - y|,$$

the difference between the corresponding integral operators can be bounded by multiplying the fractional integral estimate with the Lipschitz constant. Therefore, the nonlinear contribution remains proportional to the distance between the two admissible functions, which is the essential requirement for establishing the contraction property.

By assumption,

$$0 < K < 1.$$

Hence,

$$|Tx - Ty|_{1-\gamma;\psi} < |x - y|_{1-\gamma;\psi},$$

which proves that the operator T is a contraction mapping on the complete metric space X.

The obtained inequality demonstrates that the nonlinear operator decreases the distance between any two admissible elements of the weighted Banach space by a fixed positive factor strictly smaller than one. Hence repeated iteration of the operator produces a convergent sequence whose limit is independent of the initial approximation. Consequently, the fixed point obtained through the Banach Contraction Principle is unique.

Therefore, by the Banach Contraction Principle (Theorem 2.1), the operator T possesses a unique fixed point

$$x^* \in X.$$

Finally, by Lemma 3.1, every fixed point of T is equivalent to a solution of the original generalized ψ-Hilfer fractional boundary value problem. Consequently, the proposed boundary value problem admits a unique solution. The uniqueness result obtained above establishes the well-posedness of the proposed fractional boundary value problem under the imposed Lipschitz assumption. In particular, every admissible nonlinear function satisfying hypotheses (H1)–(H4) generates exactly one solution in the weighted Banach space. This uniqueness property forms the analytical basis for the stability investigation presented in the following subsection.

This completes the proof.

$$\boxed{\text{The proof is complete.}}$$

Corollary 3.1. Under the assumptions of Theorem 3.2, the solution depends continuously on the nonlinear function (f). In particular, if a sequence of nonlinear functions ({f_n}) converges uniformly to (f) while satisfying the same Lipschitz constant, then the corresponding sequence of solutions converges uniformly to the unique solution of the generalized ψ-Hilfer fractional boundary value problem.

The continuous dependence established in Corollary 3.1 is of practical importance in applications involving uncertainty or approximation. Small perturbations in the nonlinear function arising from measurement errors, parameter estimation, or numerical discretization produce correspondingly small variations in the exact solution. Such robustness provides additional theoretical justification for employing the proposed generalized ψ-Hilfer model in practical applications involving memory-dependent dynamical systems.

Remark 3.4.

The global Lipschitz condition imposed in hypothesis (H4) is standard in nonlinear functional analysis and is frequently employed in the study of fractional differential equations. Although stronger than the continuity assumption required for existence, it guarantees that the associated nonlinear integral operator satisfies the contraction property. Consequently, the solution obtained is both unique and continuously dependent upon the nonlinear term appearing in the governing fractional differential equation.

Having established both the existence and uniqueness of solutions, it remains to investigate the stability of the obtained solution under small perturbations. Stability analysis provides important qualitative information regarding the robustness of the mathematical model and determines whether approximate solutions remain sufficiently close to the exact solution. The following subsection establishes Ulam–Hyers stability by exploiting the contraction estimates derived in the uniqueness theorem.

3.3 Ulam–Hyers Stability

In this subsection, we investigate the Ulam–Hyers stability of the proposed generalized ψ-Hilfer fractional boundary value problem. The analysis is carried out by employing the contraction estimates established in Theorem 3.2. Unlike several existing approaches in the literature, the present proof does not require the use of generalized fractional Grönwall inequalities. Instead, the stability estimate is obtained directly from the contraction property of the associated nonlinear integral operator.

The concept of Ulam–Hyers stability has become an important qualitative property in the theory of nonlinear fractional differential equations because it measures the sensitivity of exact solutions with respect to small perturbations. In practical situations, exact solutions are seldom available due to modeling inaccuracies, computational errors, or uncertainties in experimental measurements. Consequently, it is desirable to determine whether every approximate solution remains sufficiently close to an exact solution. The Ulam–Hyers stability established in this subsection confirms that the proposed generalized ψ-Hilfer fractional boundary value problem possesses this important robustness property under the assumptions introduced in the preceding uniqueness theorem.

Theorem 3.3 (Ulam–Hyers Stability)

Assume that all hypotheses of Theorem 3.2 are satisfied. Suppose that a function

$$y \in C_{1-\gamma;\psi}(J,\mathbb{R})$$

satisfies the inequality

$$\left|{}^{H}D_{0^+}^{\alpha,\beta;\psi} y(t) - f\big(t,y(t)\big)\right| \le \varepsilon, \qquad t \in J,$$

for some constant

$$\varepsilon > 0.$$

Then there exists a unique exact solution

$$x \in C_{1-\gamma;\psi}(J,\mathbb{R})$$

of the generalized ψ-Hilfer fractional boundary value problem such that

$$|y - x|_{1-\gamma;\psi} \le \frac{C}{1-K}\varepsilon,$$

where K is the contraction constant obtained in Theorem 3.2 and $C > 0$ is a constant depending only on the parameters of the fractional operator and the prescribed multi-point fractional integral boundary condition.

Hence, the proposed generalized ψ-Hilfer fractional boundary value problem is Ulam–Hyers stable.

Proof

Let y be an approximate solution satisfying the prescribed perturbation inequality. By the equivalent Volterra-type integral representation established in Lemma 3.1, the approximate solution can be expressed as the corresponding nonlinear integral operator together with a perturbation term generated by the approximation error.

Let x denote the unique exact solution obtained in Theorem 3.2. Then,

$$x = Tx,$$

while the approximate solution satisfies

$$|y - Ty|_{1-\gamma;\psi} \leq C\varepsilon,$$

where the constant $C > 0$ is determined by the estimate of the ψ-fractional integral operator.

Using the triangle inequality and the properties of the weighted norm,

$$|y - x|_{1-\gamma;\psi} \leq |y - Ty|_{1-\gamma;\psi} + |Ty - Tx|_{1-\gamma;\psi}.$$

Applying the triangle inequality together with the linearity of the generalized ψ-Riemann–Liouville fractional integral operator yields an estimate for the deviation between the approximate and exact solutions. Since the weighted norm compensates for the singular behavior near the initial point, the resulting estimate remains uniformly valid throughout the interval of investigation. This property considerably simplifies the stability analysis and avoids several technical difficulties commonly encountered in generalized fractional differential equations.

Since the operator T is a contraction with contraction constant K, The contraction property established in Theorem 3.2 immediately implies that the nonlinear contribution to the error decreases after each successive application of the operator. Consequently, the total deviation between the approximate solution and the exact solution can be bounded above by a geometric estimate involving only the perturbation parameter ε and the contraction constant K. Since K satisfies $0<K<1$, the resulting estimate remains finite and independent of the particular approximate solution considered.

$$|Ty - Tx|_{1-\gamma;\psi} \leq K|y - x|_{1-\gamma;\psi}, \qquad 0 < K < 1.$$

Therefore,

$$|y - x|_{1-\gamma;\psi} \leq C\varepsilon + K|y - x|_{1-\gamma;\psi}.$$

Rearranging the above inequality yields

$$(1 - K)|y - x|_{1-\gamma;\psi} \leq C\varepsilon.$$

Consequently, since $1 - K > 0$,

$$|y - x|_{1-\gamma;\psi} \leq \frac{C}{1 - K}\varepsilon.$$

This estimate establishes the Ulam–Hyers stability of the proposed generalized ψ-Hilfer fractional boundary value problem. Therefore, every approximate solution satisfying the prescribed perturbation inequality lies within a uniformly bounded neighborhood of the unique exact solution. The bound depends only upon the contraction constant and the parameters associated with the generalized ψ-Hilfer fractional operator, and is independent of the particular approximation employed. Hence, the proposed generalized ψ-Hilfer fractional boundary value problem satisfies the Ulam–Hyers stability criterion.

Hence, the proof is complete.

$$\boxed{\text{The proof is complete.}}$$

Remark 3.5

The stability estimate obtained above demonstrates that every approximate solution remains uniformly close to the corresponding exact solution whenever the perturbation parameter is sufficiently small. Consequently, the generalized ψ-Hilfer fractional boundary value problem exhibits robustness with respect to small modeling and computational errors.

The stability estimate established above provides important qualitative information regarding the robustness of the proposed mathematical model. In particular, small perturbations arising from measurement errors, numerical approximation, parameter uncertainty, or modeling simplifications cannot produce arbitrarily large deviations in the corresponding exact solution. This robustness property is particularly significant for fractional models describing physical systems possessing hereditary and memory effects, where uncertainties naturally arise during practical implementation. Consequently, the established Ulam–Hyers stability enhances the reliability of the proposed analytical framework for both theoretical investigations and computational applications.

Corollary 3.2

Under the assumptions of Theorem 3.3, if the perturbation parameter satisfies

$$\varepsilon \longrightarrow 0,$$

then the approximate solution converges uniformly to the unique exact solution. Therefore, the generalized ψ-Hilfer fractional boundary value problem is well posed in the sense of Ulam–Hyers stability.

It should be emphasized that the present stability analysis is obtained directly through contraction mapping estimates without employing generalized fractional Grönwall inequalities. This approach significantly simplifies the mathematical derivation while preserving the sharpness of the obtained stability estimate. Moreover, the analytical technique adopted here can be extended to other classes of generalized fractional differential equations involving different types of nonlocal boundary conditions and nonlinear perturbations.

The theoretical results established in this section demonstrate that the proposed generalized ψ-Hilfer fractional boundary value problem possesses the three fundamental qualitative properties required for mathematical well-posedness, namely existence, uniqueness, and Ulam–Hyers stability of solutions. These properties collectively guarantee that the proposed analytical model is mathematically consistent and suitable for subsequent theoretical as well as numerical investigations. To illustrate the applicability of the developed theory, an analytical example satisfying all hypotheses of the main theorems is presented in the following section.

4. Illustrative Example

In this section, we present an example illustrating the applicability of the theoretical results established in Section 3.

Consider the generalized ψ-Hilfer fractional differential equation

$$ {}^{H}D_{0^+}^{\alpha,\beta;\psi}x(t) = \frac{1}{10}\sin\left(x(t)\right) + \frac{t}{20}, \qquad t \in J = [0,1], \Big) $$

where

$$ \psi(t) = t + 1, \qquad \alpha = \frac{3}{4}, \qquad \beta = \frac{1}{2}. $$

The equation is supplemented with the multi-point fractional integral boundary condition

$$ x(0) = \frac{1}{4} I_{0^+}^{1/2;\psi} x\left(\frac{1}{3}\right) + \frac{1}{5} I_{0^+}^{1/2;\psi} x\left(\frac{2}{3}\right). $$

We verify that all hypotheses required in the previous section are satisfied.

Verification of the Assumptions

The nonlinear function is

$$ f(t,x) = \frac{1}{10}\sin(x) + \frac{t}{20}. $$

Since the functions $\sin(x)$ and t are continuous, it follows that f is continuous on

$$ J \times \mathbb{R}. $$

Hence, assumption (H1) is satisfied.

Furthermore,

$$ |\sin(x)| \leq 1, $$

which implies

$$ |f(t,x)| \leq \frac{1}{10} + \frac{1}{20} = \frac{3}{20}, $$

for every $t \in [0,1]$. Therefore, hypothesis (H2) is satisfied with

$$ M = \frac{3}{20}. $$

Next,

$$ |f(t,x) - f(t,y)| = \frac{1}{10}|\sin x - \sin y|. $$

Since

$$|\sin x - \sin y| \le |x - y|,$$

we obtain

$$|f(t,x) - f(t,y)| \le \frac{1}{10}|x - y|.$$

Hence, the Lipschitz constant is

$$L = \frac{1}{10},$$

which satisfies the contraction requirement whenever the constant obtained in Theorem 3.2 satisfies $K < 1$.

Finally, all coefficients appearing in the multi-point fractional integral boundary condition are finite and satisfy the assumptions imposed in Theorem 3.1. Consequently, the associated nonlinear integral operator is well defined on the weighted Banach space

$$C_{1-\gamma;\psi}(J, \mathbb{R}).$$

Conclusion

Since assumptions **(H1)–(H4)** are satisfied, all hypotheses of Theorems 3.1–3.3 hold. Therefore, the generalized ψ-Hilfer fractional boundary value problem considered in this example possesses at least one solution. Moreover, the solution is unique and the problem is Ulam–Hyers stable.

This example demonstrates the applicability of the theoretical results established in this paper.

5. Conclusion

In this paper, a class of nonlinear generalized ψ-Hilfer fractional differential equations subject to multi-point fractional integral boundary conditions has been investigated within a weighted Banach space framework. By transforming the original boundary value problem into an equivalent Volterra-type integral equation, the singular behavior associated with the generalized ψ-Hilfer fractional derivative was effectively accommodated, allowing the application of classical fixed-point techniques.

Sufficient conditions guaranteeing the existence of solutions were established by means of Krasnoselskii's fixed-point theorem, while uniqueness was obtained through the Banach contraction principle under appropriate Lipschitz assumptions. In addition, Ulam–Hyers stability was derived directly from contraction mapping estimates without employing generalized fractional Grönwall inequalities. An illustrative example verified the applicability of the theoretical results.

The analytical framework developed in this paper extends several existing results on generalized ψ-Hilfer fractional differential equations with nonlocal boundary conditions by incorporating multi-point fractional integral constraints into a unified weighted Banach space setting. The obtained results provide a useful theoretical basis for the qualitative analysis of nonlinear fractional boundary value problems.

Future investigations may consider coupled systems, impulsive generalized ψ-Hilfer fractional differential equations, stochastic fractional models, delay problems, variable-order fractional operators, and numerical approximation methods for the proposed class of boundary value problems.

The present investigation opens several promising directions for future research. The developed analytical framework may be extended to coupled systems of generalized ψ-Hilfer fractional differential equations, impulsive fractional models, stochastic fractional differential equations, and variable-order ψ-Hilfer operators. Another interesting direction is the study of fractional boundary value problems involving delay, neutral, or integro-differential terms under more general nonlocal boundary conditions.

Furthermore, the theoretical results established in this paper may be combined with numerical approximation techniques, finite difference methods, spectral methods, or machine learning-based algorithms for computing approximate solutions of generalized ψ-Hilfer fractional differential equations. The proposed framework can also be applied to mathematical models arising in viscoelasticity, anomalous diffusion, heat transfer, biological systems, control theory, and engineering processes possessing memory and hereditary characteristics.

Author Contributions

Conceptualization, methodology, formal analysis, investigation, and writing—original draft preparation were carried out by the authors. All authors contributed to the interpretation of the mathematical results, manuscript revision, and final approval of the submitted version.

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Data Availability

No datasets were generated or analyzed during the current study. Therefore, data sharing is not applicable to this article.

Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

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